

# COMPUTING CENTRAL VALUES OF TWISTED $L$ -SERIES THE CASE OF COMPOSITE LEVELS

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## 1. INTRODUCTION

Let  $f \in S_2(N)$  be a newform of weight two and level  $N$ . If  $f(z) = \sum_{m=1}^{\infty} a(m) q^m$  where  $q = e^{2\pi iz}$ , and  $D$  is a fundamental discriminant, we define the twisted  $L$ -function

$$L(f, D, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \left( \frac{D}{m} \right) .$$

We will assume that the twisted  $L$ -series are primitive (i.e. the corresponding twisted modular forms are newforms). There is no loss of generality in making this assumption: if this is not the case, then  $f$  would be a quadratic twist of a newform of smaller level, which we can choose instead.

The question of efficiently computing the family of central values  $L(f, D, 1)$ , for fundamental discriminants  $D$ , has been considered by several authors. By Waldspurger's formula [Wa] these values are related to the Fourier coefficients of certain modular forms of weight  $3/2$ .

In [Gr], Gross gives a method to construct, for the case of prime level  $p$ , and provided  $L(f, 1) \neq 0$ , a weight  $3/2$  modular form of level  $4p$ , and gives an explicit version of Waldspurger's formula for the imaginary quadratic twists. In [Bö-SP] Böcherer and Schulze-Pillot extend Gross's method to the case of square free level, but their method works only for a fraction of imaginary quadratic twists (determined by quadratic residue conditions). Later in [Pa-To1] the case of level  $p^2$  ( $p$  a prime) is considered, and this is used in [Pa-To2], provided  $p \equiv 3 \pmod{4}$ , to compute central values for *real* quadratic twists.

In [MRVT] the non-vanishing condition is removed, and two modular forms of weight  $3/2$  (one giving the imaginary quadratic twists and another one giving the real quadratic twists) are constructed, in the case of prime level.

The aim of this paper is to show how some of these ideas can be combined to handle the case of composite levels. In the case of odd squarefree level  $N$ , for instance, this method constructs  $2^t$  modular forms, where  $t$  is the number of prime factors of  $N$ , whose coefficients give the central values of all the quadratic twists. We will focus on examples for levels  $N = 27$ ,  $N = 15$ , and  $N = 75$ , which exhibit most aspects of our methods.

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$D$	$c(D)$	$L(f, D, 1)$	$D$	$c(D)$	$L(f, D, 1)$	$D$	$c(D)$	$L(f, D, 1)$
-4	1	1.529954	-67	-1	0.373827	-139	3	2.335842
-7	-1	1.156537	-79	1	0.344267	-148	1	0.251523
-19	-1	0.701991	-88	-2	1.304749	-151	-1	0.249012
-31	0	0.000000	-91	1	0.320766	-163	-1	0.239670
-40	-2	1.935256	-103	1	0.301502	-184	2	0.902318
-43	2	1.866526	-115	-2	1.141352	-187	-2	0.895051
-52	1	0.424333	-127	-2	1.086092	-199	-3	1.952200
-55	2	1.650392	-136	2	1.049540			

TABLE 1. Coefficients of  $g$  and imaginary quadratic twists of 27A

## 2. THE CURVE 27A

Let  $f$  be the modular form of level 27, corresponding to the elliptic curve  $X_0(27)$ , of minimal equation

$$y^2 + y = x^3 - 7.$$

The eigenvalue of  $f$  for the Atkin-Lehner involution  $W_{27}$  is  $-1$ , and the sign of the functional equation for  $L(f, s)$  is  $+1$ .

Let  $B = (-1, -3)$  be the quaternion algebra ramified at 3 and  $\infty$ , and consider the order  $R = \langle 1, 3i, \frac{1+3j}{2}, \frac{i+k}{2} \rangle$ , a Pizer order of reduced discriminant 27. The class number of left  $R$ -ideals for such order is 2, and representatives for left  $R$ -ideals are  $\{R, I\}$  where  $I = \langle 4, 12i, \frac{7+6i+3j}{2}, \frac{6+13i+k}{2} \rangle$ . The eigenvector for the Brandt matrices which corresponds to  $f$  is  $(1, -1)$ , with height 3.

The ternary quadratic forms associated to their right orders are

$$Q_1(x, y, z) = 4x^2 + 27y^2 + 28z^2 - 4xz,$$

and

$$Q_2(x, y, z) = 7x^2 + 16y^2 + 31z^2 + 16yz + 2xz + 4xy,$$

respectively.

Note that, since the twist of  $f$  by the quadratic character of conductor 3 is  $f$  itself, we have

$$L(f, -3D, s) = L(f, D, s),$$

for  $-3D$  a fundamental discriminant. We will thus assume that  $3 \nmid D$ .

**2.1. Imaginary quadratic twists.** Let  $D < 0$  be a fundamental discriminant. If  $(\frac{D}{3}) = +1$ , the sign of the functional equation for  $L(f, D, s)$  is  $-1$ , so its central value vanishes trivially. Hence we can restrict to the case where  $(\frac{D}{3}) = -1$ . In this case we can follow Gross's method, using classical theta series

$$\Theta(Q_i) := \frac{1}{2} \sum_{(x,y,z) \in \mathbb{Z}^3} q^{Q_i(x,y,z)};$$

we obtain a weight  $3/2$  modular form of level  $4 \cdot 27$ , namely

$$g = \Theta(Q_1) - \Theta(Q_2) = q^4 - q^7 - q^{19} + q^{28} - 2q^{40} + 2q^{43} + \dots$$

Table 1 shows the values of the Fourier coefficients  $c(D)$  of  $g$  and of  $L(f, D, 1)$ ,

where  $-200 < D < 0$  is a fundamental discriminant such that  $\left(\frac{D}{3}\right) = -1$ . The Gross type formula

$$L(f, D, 1) = k \frac{|c(D)|^2}{\sqrt{|D|}}, \quad D < 0,$$

is satisfied, where  $c(D)$  is the  $|D|$ -th Fourier coefficient of  $g$ , and

$$k = \frac{1}{3} \cdot \frac{(f, f)}{L(f, 1)} = 2L(f, -4, 1) \approx 3.059908074114385749826388345.$$

**2.2. Real quadratic twists.** Let  $D > 0$  be a fundamental discriminant. In this case, if  $\left(\frac{D}{3}\right) = -1$  the sign of the functional equation for  $L(f, D, s)$  will be  $-1$ , and its central value will vanish trivially. For  $\left(\frac{D}{3}\right) = +1$ , we will employ a method similar to the one used in [MRVT] for prime levels. We need to choose an auxiliary prime  $l \equiv 3 \pmod{4}$  such that  $\left(\frac{-l}{3}\right) = -1$  and such that  $L(f, -l, 1) \neq 0$ , for example  $l = 7$ . Following [MRVT] we define generalized theta series

$$\Theta_{-7}(Q_i) := \frac{1}{2} \sum_{(x, y, z) \in \mathbb{Z}^3} \omega_7^{(i)}(x, y, z) \omega_3^{(i)}(x, y, z) q^{Q_i(x, y, z)/7},$$

where  $\omega_7$  and  $\omega_3$  are the two kinds of weight function introduced in §2.2 and §2.3 of [MRVT], respectively. The superscript in  $\omega_3^{(i)}$  and  $\omega_7^{(i)}$  indicates that we are writing the weight functions in the basis corresponding to the quadratic form  $Q_i$ .

The weight function of the first kind can be computed as

$$\omega_7^{(1)}(x, y, z) = \begin{cases} 0 & \text{if } 7 \nmid Q_1(x, y, z), \\ \left(\frac{x}{7}\right) & \text{if } 7 \nmid x, \\ \left(\frac{5z}{7}\right) & \text{otherwise;} \end{cases}$$

and

$$\omega_7^{(2)}(x, y, z) = \begin{cases} 0 & \text{if } 7 \nmid Q_2(x, y, z), \\ \left(\frac{3y+5z}{7}\right) & \text{if } 7 \nmid 3y+5z, \\ \left(\frac{6x}{7}\right) & \text{otherwise.} \end{cases}$$

The weight function of the second kind can be computed as

$$\omega_3^{(1)}(x, y, z) = \left(\frac{x+z}{3}\right), \quad \text{and} \quad \omega_3^{(2)}(x, y, z) = \left(\frac{2x+y+2z}{3}\right).$$

The generalized theta series will be

$$\Theta_{-7}(Q_1) = -2q^4 + 2q^{13} + 4q^{16} - 4q^{25} + 2q^{28} - 2q^{37} - 4q^{40} + \cdots,$$

and

$$\Theta_{-7}(Q_2) = q - q^4 - q^{13} + 2q^{16} - 3q^{25} - q^{28} + q^{37} + 2q^{40} + \cdots.$$

Note that  $\Theta_{-7}(Q_1) + 2\Theta_{-7}(Q_2) = 2q - 4q^4 + 8q^{16} - 10q^{25} + \cdots$ , corresponding to the Eisenstein eigenvector for the Brandt matrices, has nonzero Fourier coefficients only at square indices. Since  $\Theta_{-7}(Q_1) + 2\Theta_{-7}(Q_2) \equiv \Theta_{-7}(Q_1) - \Theta_{-7}(Q_2) \pmod{3}$ , this explains the fact that the coefficients in Table 2, with the exception of  $c_{-7}(1)$ , are all divisible by 3.

Thus we obtain a modular form of weight  $3/2$ , namely

$$g_{-7} = \Theta_{-7}(Q_1) - \Theta_{-7}(Q_2) = q + q^4 - 3q^{13} - 2q^{16} + q^{25} - 3q^{28} + 3q^{37} + 6q^{40} + \cdots,$$

$D$	$c_{-7}(D)$	$L(f, D, 1)$	$D$	$c_{-7}(D)$	$L(f, D, 1)$	$D$	$c_{-7}(D)$	$L(f, D, 1)$
1	1	0.588880	76	-3	0.607942	136	6	1.817856
13	-3	1.469932	85	0	0.000000	145	6	1.760536
28	-3	1.001590	88	-6	2.259892	157	6	1.691917
37	3	0.871301	97	-3	0.538125	172	0	0.000000
40	6	3.351961	109	0	0.000000	181	-9	3.545457
61	3	0.678585	124	6	1.903786	184	-6	1.562860
73	-3	0.620308	133	-3	0.459561	193	3	0.381496

TABLE 2. Coefficients of  $g_{-7}$  and real quadratic twists of  $27A$ 

and the formula is now

$$L(f, D, 1) = k_{-7} \frac{|c_{-7}(D)|^2}{\sqrt{|D|}} \quad , \quad D > 0 \quad ,$$

where  $c_{-7}(D)$  is the  $D$ -th Fourier coefficient of  $g_{-7}$ , and

$$k_{-7} = \frac{1}{3} \cdot \frac{(f, f)}{L(f, -7, 1)\sqrt{7}} = L(f, 1) \approx 0.5888795834284833191045631668 \quad .$$

Table 2 shows the values of the Fourier coefficients  $c_{-7}(D)$  of  $g_{-7}$  and of  $L(f, D, 1)$ , where  $0 < D < 200$  is a fundamental discriminant such that  $(\frac{D}{3}) = 1$ .

### 3. THE CURVE $15A$

Let  $f$  be the modular form of level 15, corresponding to the elliptic curve  $X_0(15)$ , of minimal equation

$$y^2 + xy + y = x^3 + x^2 - 10x - 10 \quad .$$

The eigenvalues of  $f$  for the Atkin-Lehner involutions  $W_3$  and  $W_5$  are  $+1$  and  $-1$ , and the sign of the functional equation for  $L(f, s)$  is  $+1$ .

The method of Gross, as extended by Böcherer and Schulze-Pillot to the case of squarefree levels, requires that the ramification of the quaternion algebra agrees with the Atkin-Lehner eigenvalues. In this case, it would be necessary to work with the quaternion algebra ramified at 5 and  $\infty$ . To exhibit the generality of our method, we will work with the quaternion algebra ramified at 3 and  $\infty$  instead.

Let  $B = (-1, -3)$  be such a quaternion algebra; an Eichler order of level 15 (index 5 in a maximal order) is given by  $R = \left\langle 1, i, \frac{1+5j}{2}, \frac{1+i+3j+k}{2} \right\rangle$ . The number of clases of left  $R$ -ideals is 2, and a set of representatives of the classes is given by  $\{R, I\}$  where  $I = \left\langle 2, 2i, \frac{3+2i+5j}{2}, \frac{3+i+3j+k}{2} \right\rangle$ . The eigenvector for the Brandt matrices corresponding to  $f$  is  $(1, -1)$ , with height 4, and the ternary quadratic forms associated to  $R$  and  $I$  are

$$Q_1(x, y, z) = Q_2(x, y, z) = 4x^2 + 15y^2 + 16z^2 - 4xz \quad .$$

**3.1. Imaginary quadratic twists.** Let  $D < 0$  be a fundamental discriminant. We say that  $D$  is of type  $(s_1, s_2)$  if  $(\frac{D}{3}) = s_1$  and  $(\frac{D}{5}) = s_2$ . We need the sign of the functional equation for  $L(f, D, s)$  to be  $+1$ , so that its central value does not vanish trivially. For this to hold we need  $D$  to be of type  $(-, +)$ ,  $(+, -)$ ,  $(+, 0)$ ,  $(0, -)$ , or  $(0, 0)$ .

$D$	$c_1(D)$	$L(f, D, 1)$	$D$	$c_1(D)$	$L(f, D, 1)$	$D$	$c_1(D)$	$L(f, D, 1)$
-4	1	1.596242	-91	-4	5.354613	-184	-4	3.765649
-19	2	2.929625	-136	-4	4.380053	-199	-2	0.905237
-31	2	2.293549	-139	-2	1.083132			
-79	-2	1.436730	-151	2	1.039203			

  

$D$	$c_{17}(D)$	$L(f, D, 1)$	$D$	$c_{17}(D)$	$L(f, D, 1)$	$D$	$c_{17}(D)$	$L(f, D, 1)$
-3	2	0.921591	-83	4	0.350421	-152	8	1.035779
-8	-4	1.128714	-87	4	0.684541	-155	8	2.051412
-15	-2	0.824296	-95	0	0.000000	-167	4	0.247042
-20	4	1.427722	-107	4	0.308629	-168	8	1.970444
-23	4	0.665679	-120	-4	1.165730	-183	0	0.000000
-35	-4	1.079257	-123	-4	0.575713	-195	-4	0.914474
-47	-4	0.465672	-132	-8	2.222961			
-68	0	0.000000	-143	-8	1.067876			

TABLE 3. Coefficients of  $g_1$  and  $g_{17}$ , and imaginary twists of  $15A$ 

Note that the linear combination of classical theta series  $\Theta(Q_1) - \Theta(Q_2)$  is trivially zero, since  $Q_1 = Q_2$ ; this reflects the fact that the ramification does not match the Atkin-Lehner eigenvalues. Instead we set

$$\Theta_1(Q_i) := \frac{1}{4} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_3^{(i)}(x, y, z) \omega_5^{(i)}(x, y, z) q^{Q_i(x,y,z)},$$

where  $\omega_3$  and  $\omega_5$  are weight functions of the second kind as in [MRVT, §2.3]. We have  $\Theta_1(Q_1) = -\Theta_1(Q_2)$ , and hence we obtain a modular form of weight  $3/2$  and level  $4 \cdot 15^2$ , namely

$$g_1 = 2 \Theta_1(Q_1) = q^4 + q^{16} + 2q^{19} + 2q^{31} + q^{64} + \dots$$

The corresponding formula is

$$L(f, D, 1) = k_1 \frac{|c_1(D)|^2}{\sqrt{|D|}}, \quad D < 0 \text{ of type } (-, +),$$

where  $c_1(D)$  is the  $|D|$ -th Fourier coefficient of  $g_1$ , and

$$k_1 = \frac{1}{4} \cdot \frac{(f, f)}{L(f, 1)} = 2L(f, -4, 1) \approx 3.192484444263567020297938143,$$

c.f. Table 3 (top).

To obtain the other 4 types of negative  $D$ , we need to choose an auxiliary prime  $l \equiv 1 \pmod{4}$  such that  $\left(\frac{l}{3}\right) = \left(\frac{l}{5}\right) = -1$ , and such that  $L(f, l, 1) \neq 0$ , e.g.  $l = 17$ . We then define the generalized theta series

$$\Theta_{17}(Q_i) := \frac{1}{4} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_{17}^{(i)}(x, y, z) q^{Q_i(x,y,z)/17},$$

where  $\omega_{17}$  is the weight function of the first kind defined in [MRVT, §2.2]. Now

$$g_{17} = 2 \Theta_{17}(Q_1) = 2q^3 - 4q^8 - 2q^{15} + 4q^{20} + 4q^{23} + \dots$$

is a weight  $3/2$  modular form of level  $4 \cdot 15$ . As expected by the multiplicity one theorem of Kohnen [Ko], this form turns out to be the same as the one constructed

$D$	$c_{-19}(D)$	$L(f, D, 1)$	$D$	$c_{-19}(D)$	$L(f, D, 1)$	$D$	$c_{-19}(D)$	$L(f, D, 1)$
1	2	0.350151	76	-16	2.570563	141	-8	0.943616
21	-8	2.445093	109	16	2.146455	156	16	3.588416
24	8	2.287175	124	16	2.012446	181	0	0.000000
61	16	2.869261	129	-8	0.986530	184	-16	1.652061
69	-8	1.348902	136	0	0.000000			
$D$	$c_{-23}(D)$	$L(f, D, 1)$	$D$	$c_{-23}(D)$	$L(f, D, 1)$	$D$	$c_{-23}(D)$	$L(f, D, 1)$
5	2	1.252737	77	-8	2.553816	152	0	0.000000
8	-4	1.980752	92	8	2.336367	173	-12	3.833492
17	4	1.358785	113	-4	0.527031	185	4	0.823795
53	4	0.769550	137	-4	0.478646	188	-8	1.634392
65	-4	1.389787	140	8	3.787922	197	12	3.592398

TABLE 4. Coefficients of  $g_{-19}$  and  $g_{-23}$ , and real twists of  $15A$ 

by Böcherer and Schulze-Pillot. The formula in this case is

$$L(f, D, 1) = \star k_{17} \frac{|c_{17}(D)|^2}{\sqrt{|D|}} \quad , \quad D < 0 \text{ of type } (+, -), (+, 0), (0, -), \text{ or } (0, 0) \quad ,$$

and  $\star = 1, 2, 2$ , or  $4$  respectively; where  $c_{17}(D)$  is the  $|D|$ -th Fourier coefficient of  $g_{17}$ , and

$$k_{17} = \frac{1}{4} \cdot \frac{(f, f)}{L(f, 17, 1)\sqrt{17}} \approx 0.1995302777664729387686211340 \quad ,$$

c.f. Table 3 (bottom).

**3.2. Real quadratic twists.** Let  $D > 0$  be a fundamental discriminant. In order for the sign of the functional equation of  $L(f, D, s)$  to be  $+1$ , we need  $D$  to be of type  $(+, +)$ ,  $(0, +)$ ,  $(-, -)$ , or  $(-, 0)$ .

For the first two types we need an auxiliary prime  $l \equiv 3 \pmod{4}$  such that  $(\frac{-l}{3}) = -1$  and  $(\frac{-l}{5}) = +1$ , and such that  $L(f, -l, 1) \neq 0$ , e.g.  $l = 19$ . Again

$$\Theta_{-19}(Q_i) := \frac{1}{4} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_{19}^{(i)}(x, y, z) \omega_5^{(i)}(x, y, z) q^{Q_i(x,y,z)/19} \quad ,$$

with  $\omega_{19}$  of the first kind and  $\omega_5$  of the second kind. The modular form

$$g_{-19} = 2 \Theta_{-19}(Q_1) = 2q - 4q^4 + 2q^9 - 8q^{21} + 8q^{24} + \dots$$

has level  $4 \cdot 15 \cdot 5$ , and the formula is

$$L(f, D, 1) = \star k_{-19} \frac{|c_{-19}(D)|^2}{\sqrt{|D|}} \quad , \quad D > 0 \text{ of type } (+, +) \text{ or } (0, +) \quad ,$$

$\star = 1$  or  $2$  respectively;  $c_{-19}(D)$  is the  $D$ -th Fourier coefficient of  $g_{-19}$ , and

$$k_{-19} = \frac{1}{4} \cdot \frac{(f, f)}{L(f, -19, 1)\sqrt{19}} = \frac{1}{4} L(f, 1) \approx 0.08753769014578762644876130241 \quad .$$

Table 4 (top) shows the values of the coefficients  $c_{-19}(D)$  and the central values  $L(f, D, 1)$  for  $0 < D < 200$  a fundamental discriminant of type  $(+, +)$  or  $(0, +)$ .

For the remaining two types we need an auxiliary prime  $l \equiv 3 \pmod{4}$  such that  $\left(\frac{-l}{3}\right) = +1$  and  $\left(\frac{-l}{5}\right) = -1$ , and such that  $L(f, -l, 1) \neq 0$ , e.g.  $l = 23$ . As before we define

$$\Theta_{-23}(Q_i) := \frac{1}{4} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_{23}^{(i)}(x, y, z) \omega_3^{(i)}(x, y, z) q^{Q_i(x,y,z)/23} ,$$

with  $\omega_{23}$  of the first kind and  $\omega_3$  of the second kind. The modular form

$$g_{-23} = 2\Theta_{-23}(Q_1) = 2q^5 - 4q^8 + 4q^{17} - 4q^{32} + 4q^{53} + \dots$$

has level  $4 \cdot 15 \cdot 3$ , and the formula is

$$L(f, D, 1) = \star k_{-23} \frac{|c_{-23}(D)|^2}{\sqrt{|D|}} , \quad D > 0 \text{ of type } (-, -) \text{ or } (-, 0) ,$$

$\star = 1$  or  $2$  respectively;  $c_{-23}(D)$  is the  $D$ -th Fourier coefficient of  $g_{-23}$  and

$$k_{-23} = \frac{1}{4} \cdot \frac{(f, f)}{L(f, -23, 1)\sqrt{23}} \approx 0.3501507605831505057950452092 .$$

Table 4 (bottom) shows the values of the coefficients  $c_{-19}(D)$  and the central values  $L(f, D, 1)$  for  $0 < D < 200$  a fundamental discriminant of type  $(-, -)$  or  $(-, 0)$ .

#### 4. THE CURVE 75A

Let  $f$  be the modular form of level 75 corresponding to the elliptic curve of minimal equation

$$y^2 + y = x^3 - x^2 - 8x - 7 .$$

The eigenvalue of  $f$  for the Atkin-Lehner involution  $W_3$  is  $+1$ , for  $W_{25}$  is  $-1$ , and the sign of the functional equation for  $L(f, s)$  is  $+1$ .

Let  $B = (-1, -3)$  be the quaternion algebra ramified at 3 and  $\infty$ , and consider the order  $R = \langle 1, i, \frac{1+5j}{2}, \frac{i+5k}{2} \rangle$ , an Eichler order of level 75 (index 25 in a maximal order). The class number of left  $R$ -ideals is 6, and the eigenvector for the Brandt matrices which corresponds to  $f$  is  $(1, -1, 1, -1, 0, 0)$ , with height 6.

The ternary quadratic forms associated to the right orders of the choosen ideal class representatives are

$$\begin{aligned} Q_1(x, y, z) &= Q_2(x, y, z) = 4x^2 + 75y^2 + 76z^2 - 4xz , \\ Q_3(x, y, z) &= Q_4(x, y, z) = 16x^2 + 19y^2 + 79z^2 + 4xy + 16xz + 2yz , \end{aligned}$$

and

$$Q_5(x, y, z) = Q_6(x, y, z) = 24x^2 + 31y^2 + 39z^2 + 24xy + 12xz + 6yz ,$$

respectively.

We will assume that  $5 \nmid D$ . Indeed, the twist of  $f$  by the quadratic character of conductor 5 is another modular form  $f'$  of level 75, thus we have

$$L(f, 5D, 1) = L(f', D, 1),$$

for  $5D$  a fundamental discriminant. By applying the same procedure to the modular form  $f'$  we can compute the central values for these twists. So, we actually need 8 different modular forms of weight  $3/2$  to compute all the twisted central values.

$D$	$c_1(D)$	$L(f, D, 1)$	$D$	$c_1(D)$	$L(f, D, 1)$	$D$	$c_1(D)$	$L(f, D, 1)$
-4	1	2.334766	-91	-1	0.489500	-184	2	1.376970
-19	-1	1.071264	-136	2	1.601637	-199	-5	8.275360
-31	-1	0.838673	-139	-2	1.584258			
-79	4	8.405816	-151	5	9.500030			
$D$	$c_{13}(D)$	$L(f, D, 1)$	$D$	$c_{13}(D)$	$L(f, D, 1)$	$D$	$c_{13}(D)$	$L(f, D, 1)$
-7	3	5.294752	-88	-6	5.973286	-163	3	1.097238
-43	3	2.136291	-103	-6	5.521233	-187	0	0.000000
-52	3	1.942643	-127	-6	4.972248			
-67	-3	1.711423	-148	0	0.000000			

TABLE 5. Coefficients of  $g_1$  and  $g_{13}$ , and imaginary twists of  $75A$ 

**4.1. Imaginary quadratic twists.** Let  $D < 0$  be a fundamental discriminant. If the sign of the functional equation for  $L(f, D, s)$  is  $+1$ , the type of  $D$  has to be either  $(-, +)$  or  $(-, -)$ .

For the first case we look at the generalized theta series

$$\Theta_1(Q_i) := \frac{1}{4} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_3^{(i)}(x, y, z) \omega_5^{(i)}(x, y, z) q^{Q_i(x,y,z)} ;$$

we obtain the modular form

$$g_1 = 2\Theta_1(Q_1) - 2\Theta_1(Q_3) = q^4 - 2q^{16} - q^{19} - q^{31} - 2q^{64} + 3q^{76} + 4q^{79} - q^{91} + \dots .$$

The formula

$$L(f, D, 1) = k_1 \frac{|c_1(D)|^2}{\sqrt{|D|}} , \quad D < 0 \text{ of type } (-, +) ,$$

is satisfied (c.f. Table 5, top), where  $c_1(D)$  is the  $|D|$ -th Fourier coefficient of  $g_1$  and

$$k_1 = \frac{1}{6} \cdot \frac{(f, f)}{L(f, 1)} = 2 L(f, -4, 1) \approx 4.669532748718719327951206761 .$$

In the second case we need to choose an auxiliary prime  $l \equiv 1 \pmod{4}$  such that  $\left(\frac{l}{3}\right) = +1$ ,  $\left(\frac{l}{5}\right) = -1$  and  $L(f, l, 1) \neq 0$ , for example  $l = 13$ , and define

$$\Theta_{13}(Q_i) := \frac{1}{4} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_{13}^{(i)}(x, y, z) \omega_3^{(i)}(x, y, z) \omega_5^{(i)}(x, y, z) q^{Q_i(x,y,z)/13} .$$

We obtain the modular form

$$g_{13} = 2\Theta_{13}(Q_1) - 2\Theta_{13}(Q_3) = 3q^7 + 3q^{28} + 3q^{43} + 3q^{52} - 3q^{67} - 6q^{88} + \dots ,$$

and the formula

$$L(f, D, 1) = k_{13} \frac{|c_{13}(D)|^2}{\sqrt{|D|}} , \quad D < 0 \text{ of type } (+, +) ,$$

is satisfied (c.f. Table 5, bottom), where  $c_{13}(D)$  is the  $|D|$ -th Fourier coefficient of  $g_{13}$  and

$$k_{13} = \frac{1}{6} \cdot \frac{(f, f)}{L(f, 13, 1)\sqrt{13}} \approx 1.556510916239573109317068920 .$$



$D$	$c_{-19}(D)$	$L(f, D, 1)$	$D$	$c_{-19}(D)$	$L(f, D, 1)$	$D$	$c_{-19}(D)$	$L(f, D, 1)$
1	1	1.402540	76	1	0.160882	141	2	0.944921
21	-1	0.612119	109	-1	0.134339	156	-1	0.224586
24	-2	2.290338	124	5	3.148795	181	3	0.938250
61	-1	0.179577	129	5	6.174338	184	-2	0.413586
69	2	1.350768	136	-6	4.329605			
12	3	2.429270	73	6	1.969859	168	6	2.596999
13	3	1.166984	88	-6	1.794135	172	3	0.320828
28	-3	0.795165	93	-3	0.872620	177	-6	2.530113
33	-6	5.859621	97	9	3.844972	193	9	2.725840
37	0	0.000000	133	-3	0.364847			
57	-3	1.114626	157	3	0.335805			

TABLE 6. Coefficients of  $g_{-19}$  and  $g_{-7}$ , and real twists of  $75A$ 

**4.2. Real quadratic twists.** Let  $D > 0$  be a fundamental discriminant. The only possibilities so that the sign of the functional equation for  $L(f, D, s)$  is  $+1$  are the discriminants  $D$  of types  $(+, +)$ ,  $(0, +)$ ,  $(+, -)$ , and  $(0, -)$ .

For the first two cases we can use the generalized theta series

$$\Theta_{-19}(Q_i) := \frac{1}{2} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_{19}^{(i)}(x, y, z) \omega_5^{(i)}(x, y, z) q^{Q_i(x,y,z)/19}.$$

Thus we obtain a modular form of weight  $3/2$ , namely

$$g_{-19} = q + q^4 + q^9 - q^{21} - 2q^{24} - q^{36} - 4q^{49} - q^{61} + \dots,$$

and the formula is

$$L(f, D, 1) = \star k_{-19} \frac{|c_{-19}(D)|^2}{\sqrt{|D|}}, \quad D > 0 \text{ of type } (+, +) \text{ or } (0, +),$$

$\star = 1$  or  $2$  respectively,  $c_{-19}(D)$  the  $D$ -th Fourier coefficient of  $g_{-19}$ , and

$$k_{-19} = \frac{1}{6} \cdot \frac{(f, f)}{L(f, -19, 1)\sqrt{19}} = L(f, 1) \approx 1.402539940216221119844494086,$$

c.f. Table 6 (top).

In the other two cases we can use the generalized theta series

$$\Theta_{-7}(Q_i) := \frac{1}{2} \sum_{(x,y,z) \in \mathbb{Z}^3} \omega_7^{(i)}(x, y, z) \omega_5^{(i)}(x, y, z) q^{Q_i(x,y,z)/7}.$$

We obtain a modular form of weight  $3/2$

$$g_{-7} = 3q^{12} + 3q^{13} - 3q^{28} - 6q^{33} + 6q^{48} - 9q^{52} - 3q^{57} + 6q^{73} + \dots,$$

satisfying the formula

$$L(f, D, 1) = \star k_{-7} \frac{|c_{-7}(D)|^2}{\sqrt{|D|}}, \quad D > 0 \text{ of type } (+, -) \text{ or } (0, -),$$

$\star = 1$  or  $2$  respectively,  $c_{-7}(D)$  the  $D$ -th Fourier coefficient of  $g_{-7}$ , and

$$k_{-7} = \frac{1}{6} \cdot \frac{(f, f)}{L(f, -7, 1)\sqrt{7}} \approx 0.4675133134054070399481646950.$$

c.f. Table 6 (bottom).

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